

Corrigé Devoir Analyse

Exercice 1.

Calculer les limites des suites suivantes :

$$a_n = \frac{\left(\frac{1}{7}\right)^n - \left(\frac{4}{5}\right)^n}{\left(\frac{1}{7}\right)^n + \left(\frac{4}{5}\right)^n} \quad b_n = \frac{n}{n^2+1} + \frac{n}{n^2+2} + \dots + \frac{n}{n^2+n} \quad u_n = \sqrt{16n^4 + 3n^2 + 6} - 4n^2 - 1$$

$$v_n = \ln(6n-1) - \ln(n+1) + \ln(2n+3) - \ln(4n+1)$$

Corrigé:

$$\bullet a_n = \frac{\left(\frac{1}{7}\right)^n - \left(\frac{4}{5}\right)^n}{\left(\frac{1}{7}\right)^n + \left(\frac{4}{5}\right)^n}$$

$$\lim_{n \rightarrow +\infty} a_n = \lim_{n \rightarrow +\infty} \frac{\left(\frac{1}{7}\right)^n - \left(\frac{4}{5}\right)^n}{\left(\frac{1}{7}\right)^n + \left(\frac{4}{5}\right)^n} = \lim_{n \rightarrow +\infty} \frac{\frac{\left(\frac{1}{7}\right)^n - \left(\frac{4}{5}\right)^n}{\left(\frac{4}{5}\right)^n}}{\frac{\left(\frac{1}{7}\right)^n + \left(\frac{4}{5}\right)^n}{\left(\frac{4}{5}\right)^n}} = \lim_{n \rightarrow +\infty} \frac{\frac{\left(\frac{1}{7}\right)^n}{\left(\frac{4}{5}\right)^n} - 1}{\frac{\left(\frac{1}{7}\right)^n}{\left(\frac{4}{5}\right)^n} + 1} = \lim_{n \rightarrow +\infty} \frac{\left(\frac{5}{28}\right)^n - 1}{\left(\frac{5}{28}\right)^n + 1}$$

$$\lim_{n \rightarrow +\infty} a_n = \frac{-1}{1} = -1$$

$$\bullet b_n = \frac{n}{n^2+1} + \frac{n}{n^2+2} + \dots + \frac{n}{n^2+n}$$

$$b_n = \sum_{k=1}^n \frac{n}{n^2+k}$$

$$1 \leq k \leq n \Rightarrow n^2+1 \leq n^2+k \leq n^2+n \Rightarrow \frac{1}{n^2+n} \leq \frac{1}{n^2+k} \leq \frac{1}{n^2+1}$$

$$\Rightarrow \sum_{k=1}^n \frac{1}{n^2+n} \leq \sum_{k=1}^n \frac{1}{n^2+k} \leq \sum_{k=1}^n \frac{1}{n^2+1} \Rightarrow \frac{n^2}{n^2+n} \leq b_n \leq \frac{n^2}{n^2+1}$$

$$\Rightarrow \lim_{n \rightarrow +\infty} \frac{n^2}{n^2+n} \leq \lim_{n \rightarrow +\infty} b_n \leq \lim_{n \rightarrow +\infty} \frac{n^2}{n^2+1} \Rightarrow 1 \leq \lim_{n \rightarrow +\infty} b_n \leq 1$$

$$\text{d'après TG } \lim_{n \rightarrow +\infty} b_n = 1$$

$$\bullet u_n = \sqrt{16n^4 + 3n^2 + 6} - 4n^2 - 1$$

$$\lim_{n \rightarrow +\infty} u_n = \lim_{n \rightarrow +\infty} \sqrt{16n^4 + 3n^2 + 6} - 4n^2 - 1$$

$$= \lim_{n \rightarrow +\infty} \frac{(\sqrt{16n^4 + 3n^2 + 6} - 4n^2 - 1)(\sqrt{16n^4 + 3n^2 + 6} + 4n^2 + 1)}{\sqrt{16n^4 + 3n^2 + 6} + 4n^2 + 1} = \lim_{n \rightarrow +\infty} \frac{16n^4 + 3n^2 + 6 - 16n^4 - 8n^2 - 1}{\sqrt{16n^4 + 3n^2 + 6} + 4n^2 + 1}$$

$$= \lim_{n \rightarrow +\infty} \frac{-5n^2 + 5}{\sqrt{16n^4 + 3n^2 + 6} + 4n^2 + 1} = \lim_{n \rightarrow +\infty} \frac{n^2(-5 + \frac{5}{n^2})}{n^2(\sqrt{16 + \frac{3}{n} + \frac{6}{n^2}} + 4 + \frac{1}{n^2})} = \frac{-5}{8}$$

$$\lim_{n \rightarrow +\infty} u_n = \frac{-5}{8}$$

$$\bullet v_n = \ln(6n - 1) - \ln(n + 1) + \ln(2n + 3) - \ln(4n + 1)$$

$$\begin{aligned} \lim_{n \rightarrow +\infty} v_n &= \lim_{n \rightarrow +\infty} [\ln(6n - 1) - \ln(n + 1) + \ln(2n + 3) - \ln(4n + 1)] \\ &= \lim_{n \rightarrow +\infty} \ln\left(\frac{6n-1}{n+1}\right) + \ln\left(\frac{2n+3}{4n+1}\right) = \ln(6) + \ln\left(\frac{2}{4}\right) = \ln\left(\frac{6 \times 2}{4}\right) = \ln(3) \end{aligned}$$

$$\lim_{n \rightarrow +\infty} v_n = \ln(3)$$

Exercice 2.

Montrer, à partir de la définition de limite, que $\lim_{n \rightarrow +\infty} \frac{5n+1}{3+n} = 5$

Corrigé:

$$\lim_{n \rightarrow +\infty} \frac{5n+1}{3+n} = 5$$

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, \left| \frac{5n+1}{3+n} - 5 \right| < \varepsilon$$

$$\begin{aligned} \left| \frac{5n+1}{3+n} - 5 \right| < \varepsilon &\Rightarrow \left| \frac{5n+1-15-5n}{3+n} \right| < \varepsilon \Rightarrow \left| \frac{-14}{3+n} \right| < \varepsilon \Rightarrow \frac{14}{3+n} < \varepsilon \Rightarrow 14 < 3\varepsilon + \varepsilon n \\ &\Rightarrow 14 - 3\varepsilon < \varepsilon n \Rightarrow \frac{14-3\varepsilon}{\varepsilon} < n \Rightarrow \frac{14}{\varepsilon} - 3 < n \end{aligned}$$

$$\text{il suffit de prendre : } N_\varepsilon = \left[\frac{14}{\varepsilon} - 3 \right]$$

$$\forall \varepsilon > 0, \exists N_\varepsilon \in \mathbb{N}, \left| \frac{5n+1}{3+n} - 5 \right| < \varepsilon, \quad \lim_{n \rightarrow +\infty} \frac{5n+1}{3+n} = 5$$

Exercice 3.

Soit $\{u_n\}$ la suite réelle définie par : $u_1 = \frac{1}{2}$, $u_{n+1} = \frac{2u_n+1}{u_n+1}$

1) Calculer u_2 et u_3

2) Montrer par récurrence que pour tout entier naturel n on a : $0 \leq u_n \leq 2$

3) Etudier la monotone de $\{u_n\}$

4) En déduire que $\{u_n\}$ converge et déterminer sa limite.

Corrigé:

$$\text{On a } u_1 = \frac{1}{2}, \quad u_{n+1} = \frac{2u_n+1}{u_n+1}$$

1)

$$\bullet u_2 = \frac{2u_1+1}{u_1+1} = \frac{1+1}{\frac{3}{2}} = \frac{4}{3}$$

$$\bullet u_3 = \frac{2u_2+1}{u_2+1} = \frac{\frac{8}{3}+1}{\frac{4}{3}+1} = \frac{11}{7}$$

2)

Pour $n = 1$ $u_1 = \frac{1}{2}$ et $0 \leq \frac{1}{2} \leq 2$

On suppose que $0 \leq u_n \leq 2$ et on montre que $0 \leq u_{n+1} \leq 2$

$$0 \leq u_n \leq 2$$

$$0 \leq u_n \leq 2$$

$$0 \leq 2u_n \leq 4$$

$$1 \leq u_n + 1 \leq 3$$

$$1 \leq 2u_n + 1 \leq 5 \quad (1)$$

$$\frac{1}{3} \leq \frac{1}{u_n + 1} \leq 1 \quad (2)$$

$$(1) \times (2): 0 < \frac{1}{3} \leq \frac{2u_n + 1}{u_n + 1} \leq 5$$

$$0 \leq u_{n+1} \leq 5$$

on calcule

$$u_{n+1} - 2 = \frac{2u_n + 1}{u_n + 1} - 2 = \frac{2u_n + 1 - 2u_n - 2}{u_n + 1} = \frac{-1}{u_n + 1} < 0 \text{ donc } u_{n+1} - 2 < 0 \Rightarrow u_{n+1} < 2$$

$$\Rightarrow 0 \leq u_{n+1} \leq 2$$

$$\text{donc } \forall n \in \mathbb{N} \quad 0 \leq u_n \leq 2$$

3)

$$u_{n+1} - u_n = \frac{2u_n + 1}{u_n + 1} - u_n = \frac{2u_n + 1 - (u_n)^2 - u_n}{u_n + 1} = \frac{-(u_n)^2 - u_n + 1}{u_n + 1}$$

$$-(u_n)^2 - u_n + 1$$

$$\Delta = 1 - 4 \times (-1) = 1 + 4 = 5$$

$$u_{n_1} = \frac{-1 - \sqrt{5}}{-2} = \frac{1 + \sqrt{5}}{2} \approx 1.6, \quad u_{n_2} = \frac{1 - \sqrt{5}}{2} \approx -0.6$$

Exercice 4.

Calculer, lorsqu'elles existent, les limites des fonctions suivantes :

$$1) \lim_{x \rightarrow 0} \left(\frac{2}{(\sin x)^2} - \frac{1}{1 - \cos x} \right) \quad 2) \lim_{x \rightarrow 1} \frac{\sqrt{3x+1} - \sqrt{x+3}}{\sqrt[3]{x} - 1} \quad 3) \lim_{x \rightarrow 0} \frac{e^{2x} + 5e^x - 6}{e^x - 1}$$

Corrigé:

$$1) \lim_{x \rightarrow 0} \left(\frac{2}{(\sin x)^2} - \frac{1}{1 - \cos x} \right)$$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{2}{(\sin x)^2} - \frac{1}{1 - \cos x} &= \lim_{x \rightarrow 0} \frac{2}{(1 - \cos x)(1 + \cos x)} - \frac{1}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{2 - (1 + \cos x)}{(1 - \cos x)(1 + \cos x)} \\ &= \lim_{x \rightarrow 0} \frac{1 - \cos x}{(1 - \cos x)(1 + \cos x)} = \lim_{x \rightarrow 0} \frac{1}{1 + \cos x} = \frac{1}{1 + 1} = \frac{1}{2} \end{aligned}$$

$$\lim_{x \rightarrow 0} \frac{2}{(\sin x)^2} - \frac{1}{1 - \cos x} = \frac{1}{2}$$

$$2) \lim_{x \rightarrow 1} \frac{\sqrt{3x+1} - \sqrt{x+3}}{\sqrt[3]{x} - 1}$$

$$\lim_{x \rightarrow 1} \frac{\sqrt{3x+1} - \sqrt{x+3}}{\sqrt[3]{x} - 1} = ?$$

$$\lim_{x \rightarrow 1} \sqrt{3x+1} - \sqrt{x+3} H = \lim_{x \rightarrow 1} \frac{3}{2\sqrt{3x+1}} - \frac{1}{2\sqrt{x+3}} = \frac{3}{2\sqrt{4}} - \frac{1}{2\sqrt{4}} = \frac{1}{2}$$

$$\lim_{x \rightarrow 1} \sqrt[3]{x} - 1 H = \lim_{x \rightarrow 1} \frac{1}{3} x^{-\frac{2}{3}} = \frac{1}{3}$$

$$\lim_{x \rightarrow 1} \frac{\sqrt{3x+1} - \sqrt{x+3}}{\sqrt[3]{x} - 1} = \frac{\lim_{x \rightarrow 1} (\sqrt{3x+1} - \sqrt{x+3})}{\lim_{x \rightarrow 1} (\sqrt[3]{x} - 1)} = \frac{\frac{1}{2}}{\frac{1}{3}} = \frac{3}{2}$$

$$\lim_{x \rightarrow 1} \frac{\sqrt{3x+1} - \sqrt{x+3}}{\sqrt[3]{x} - 1} = \frac{3}{2}$$

$$3) \lim_{x \rightarrow 0} \frac{e^{2x} + 5e^x - 6}{e^x - 1}$$

$$\lim_{x \rightarrow 0} \frac{e^{2x} + 5e^x - 6}{e^x - 1} = \lim_{x \rightarrow 0} \frac{(e^x - 1)(e^x + 6)}{e^x - 1} = \lim_{x \rightarrow 0} e^x + 6 = e^0 + 6 = 7$$

$$\lim_{x \rightarrow 0} \frac{e^{2x} + 5e^x - 6}{e^x - 1} = 7$$

Exercice 5.

1) Etudier la continuité des fonctions suivantes :

$$f(x) = \frac{\sqrt{x^2+5x+7}-1}{x+2} \text{ si } x \neq -2, \frac{1}{2} \text{ si } x = -2$$

$$g(x) = \ln(1+x^2) \sin\left(\frac{1}{1+x^2}\right) \text{ si } x \neq 0, 1 \text{ si } x = 0$$

2)

Peut-on prolonger la fonction suivante par continuité au point 1? Si oui, donner l'expression du prolongement.

$$f(x) = \frac{x^3 - 2x^2 - x + 2}{x-1}$$

Corrigé:

1)

$$\bullet f(x) = \frac{\sqrt{x^2+5x+7}-1}{x+2} \text{ si } x \neq -2, \frac{1}{2} \text{ si } x = -2$$

$$\begin{aligned} \lim_{x \rightarrow -2} f(x) &= \lim_{x \rightarrow -2} \frac{\sqrt{x^2+5x+7}-1}{x+2} = \lim_{x \rightarrow -2} \frac{(\sqrt{x^2+5x+7}-1)(\sqrt{x^2+5x+7}+1)}{x+2(\sqrt{x^2+5x+7}+1)} \\ &= \lim_{x \rightarrow -2} \frac{x^2+5x+6}{x+2(\sqrt{x^2+5x+7}+1)} = \lim_{x \rightarrow -2} \frac{x^2+5x+6}{x+2(\sqrt{x^2+5x+7}+1)} = \lim_{x \rightarrow -2} \frac{(x+2)(x+3)}{x+2(\sqrt{x^2+5x+7}+1)} \end{aligned}$$

$$= \lim_{x \rightarrow -2} \frac{x+3}{\sqrt{x^2+5x+7}+1} = \frac{-2+3}{\sqrt{(-2)^2+5(-2)+7}+1} = \frac{1}{2}$$

$$\lim_{x \rightarrow -2} f(x) = \frac{1}{2}$$

comme $\lim_{x \rightarrow -2} f(x) = f(-2)$ donc la fonction f est continue en -2

• $g(x) = \ln(1 + x^2) \sin\left(\frac{1}{1+x^2}\right)$ si $x \neq 0$, 1 si $x = 0$

$$\begin{aligned} \lim_{x \rightarrow 0} g(x) &= \lim_{x \rightarrow 0} \left(\ln(1 + x^2) \sin\left(\frac{1}{1+x^2}\right) \right) = \lim_{x \rightarrow 0} \ln(1 + x^2) \times \lim_{x \rightarrow 0} \sin\left(\frac{1}{1+x^2}\right) \\ &= \ln(1) \times \lim_{x \rightarrow 0} \sin\left(\frac{1}{1+x^2}\right) = 0 \times \lim_{x \rightarrow 0} \sin\left(\frac{1}{1+x^2}\right) = 0 \end{aligned}$$

$$\lim_{x \rightarrow 0} g(x) = 0$$

comme $\lim_{x \rightarrow 0} g(x) = g(0)$ donc la fonction f n'est pas continue en 0

